Lecture 23

Multinomial Theorem and Principle of Inclusion-Exclusion

Multinomial Theorem

Multinomial coefficients

Multinomial Theorem: For all non-negative integer n and k,

$$(x_1 + x_2 + \ldots + x_k)^n = \sum_{a_1, a_2, \ldots, a_k} \binom{n}{a_1, a_2, \ldots, a_k} x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k}$$

where the sum is taken over all k-tuples of $a_1, a_2, \ldots, a_k$ such that $a_i \in \mathbb{N}$ and $n = \sum_{i=1}^{k} a_i$.

Relating Binomial and Multinomial Coefficients

 # different positions type 2 objects can take in linear ordering of n objects after type 1 objects are placed.

 # different positions type k objects can take in linear ordering of n objects after other types of objects are placed.

Theorem: For all non-negative integers n and $a_1, a_2, \ldots, a_k$ such that $n = \sum_{i=1}^{k} a_i$.,

$$\binom{n}{a_1, a_2, \ldots, a_k} = \binom{n}{a_1}\binom{n - a_1}{a_2}\binom{n - a_1 - a_2}{a_3} \cdots\cdots \binom{n - a_1 - a_2 - \ldots - a_{k-1}}{a_k}$$

 # linear ordering of a multiset of n objects where a_i items are of type i.

 # different positions type 1 objects can take in linear ordering of n objects.

 # different positions type 3 objects can take in linear ordering of n objects after type 1 & 2 objects are placed.

Multinomial Theorem

Multinomial Theorem: For all non-negative integer n and k,

$$(x_1 + x_2 + \ldots + x_k)^n = \sum_{a_1, a_2, \ldots, a_k} \binom{n}{a_1, a_2, \ldots, a_k} x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k}$$

where the sum is taken over all k-tuples of $a_1, a_2, \ldots, a_k$ such that $a_i \in \mathbb{N}$ and $n = \sum_{i=1}^{k} a_i$.

Proof: We will prove that $x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k}$ occurs $\binom{n}{a_1, a_2, \ldots, a_k}$ times in the expansion of $(x_1 + x_2 + \ldots + x_k)^n$.

To get $x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k}$ we have to choose x_i from exactly a_i parentheses.

Multinomial Theorem

times $x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k}$ occurs in the expansion of $(x_1 + x_2 + \ldots + x_k)^n$

$$=$$

ways x_1 can be picked from a_1 parentheses out of n parentheses. $\times$ # ways x_2 can be picked from a_2 parentheses out of $n - a_1$ parentheses. $\times \ldots \times$ # ways x_k can be picked from a_k parentheses out of $n - \sum_{i=1}^{k-1} a_i$ parentheses.

$$=$$

$$\binom{n}{a_1} \times \binom{n - a_1}{a_2} \times \ldots \times \binom{n - \sum_{i=1}^{k-1} a_i}{a_k}$$

$$=$$

$$\binom{n}{a_1, a_2, \ldots, a_k}$$

■

Idea: Principle of Inclusion-Exclusion

There are 14 students in a high school class who play soccer, and there are 17 students who play basketball.

How many students play soccer or basketball?

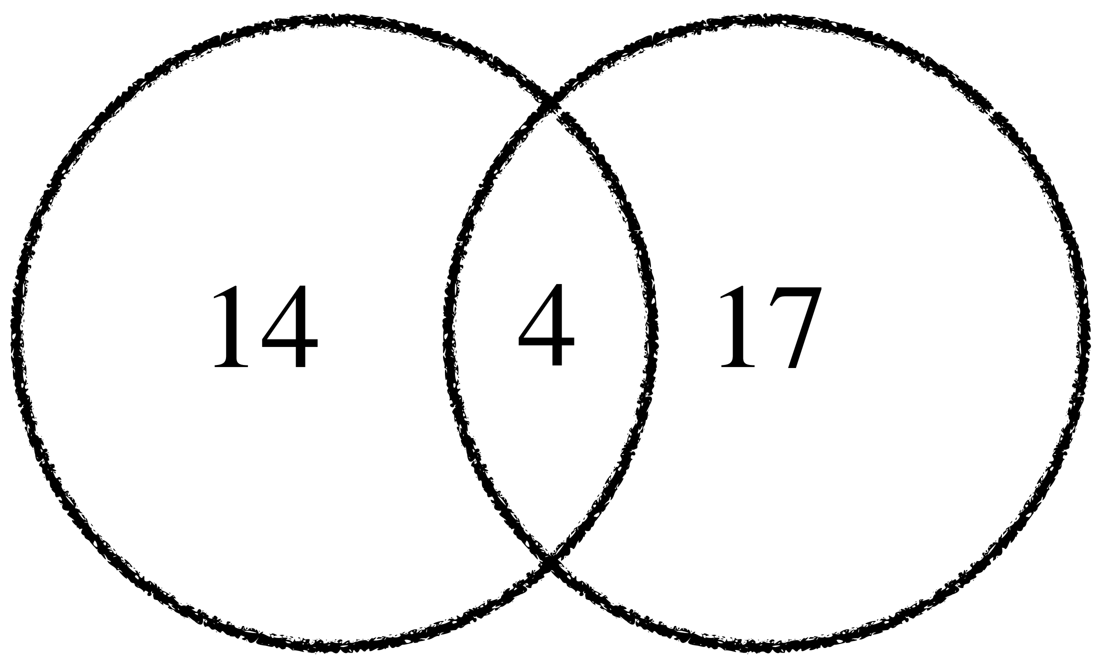
$14 + 17$ might not be the right answer because some students might be playing both sports.

Suppose 4 students play both the sports. Then,

$$\text{\# students who play soccer or basketball} = 14 + 17 - 4$$

Subtracting the overcounted.

Soccer

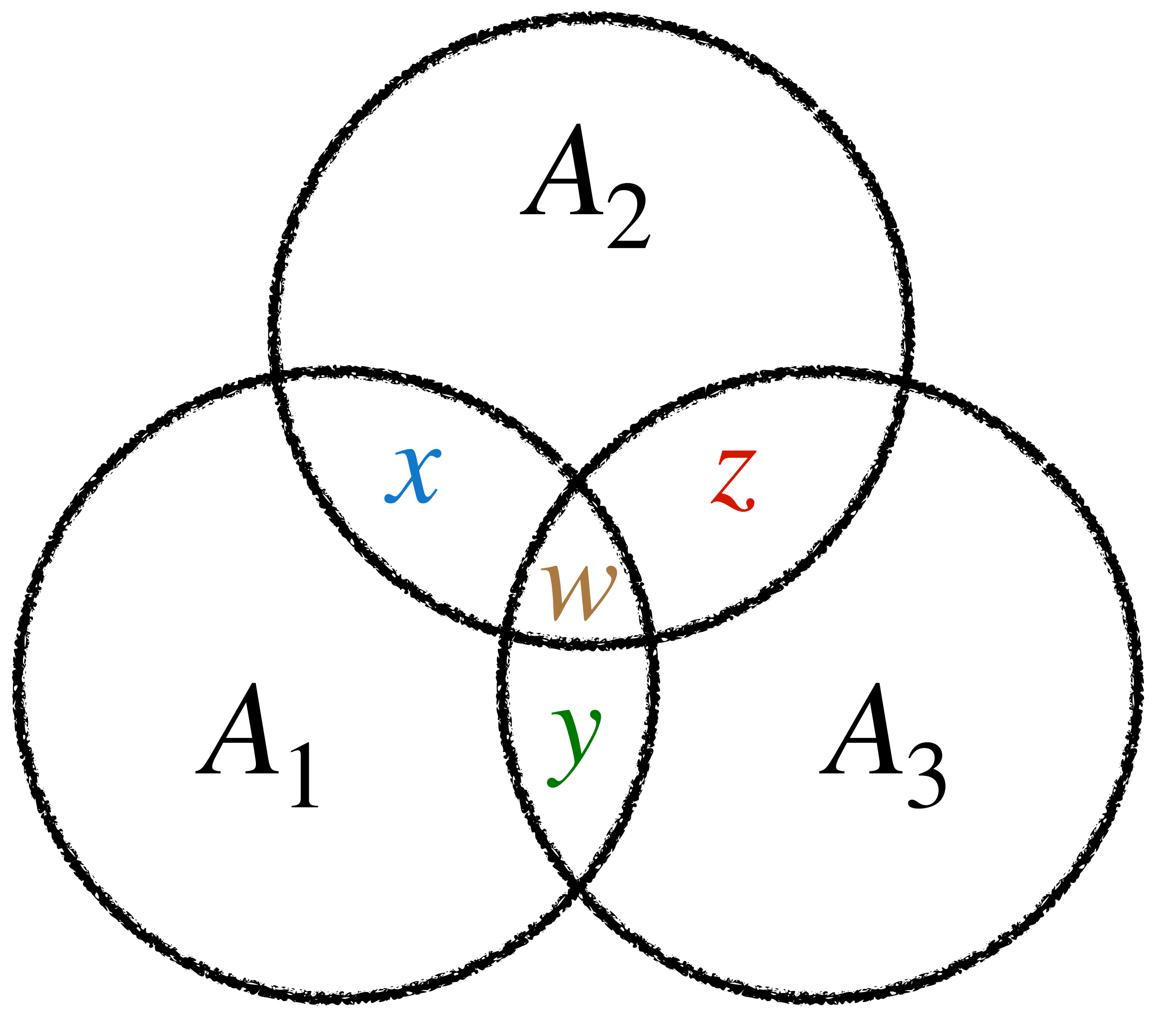


Basketball

Idea: Principle of Inclusion-Exclusion

Early instances of principle of inclusion-exclusion:

- $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
- $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_1|$
  $+ |A_1 \cap A_2 \cap A_3|$



Principle of Inclusion-Exclusion

Theorem: Let $A_1, A_2, \ldots, A_n$ be the finite sets. Then,

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{j=1}^{n} (-1)^{j-1} \sum_{i_1, i_2, \ldots, i_j} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_j}|$$

Proof: We show that each element in $A_1 \cup A_2 \cup \ldots \cup A_n$ is counted exactly once on the RHS.

Let $x \in A_1 \cup A_2 \cup \ldots \cup A_n$.

Let $S = \{i_1, i_2, \ldots, i_s\}$ such that $x \in A_i$ iff $i \in S$.

Observation: $A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}$ contains x if and only if $\{i_1, i_2, \ldots, i_k\} \subseteq S$.

Principle of Inclusion-Exclusion

x is counted or subtracted the following times on the RHS:

- Counted s times in single intersections.
- Subtracted $\binom{s}{2}$ times in double intersections.
- Counted $\binom{s}{3}$ times in triple intersections.

$$\vdots$$

So, RHS counts x

$$s - \binom{s}{2} + \binom{s}{3} - \binom{s}{4} \cdots + (-1)^{s-1}\binom{s}{s} = 1$$

■